Photon Filamentation in Resonant Media with High Fresnel Numbers

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The phenomenon of turbulent photon filamentation occurs in lasers and other active optical media at high Fresnel numbers. A description of this phenomenon is suggested. The solutions to evolution equations are presented in the form of a bunch of filaments chaotically distributed in space and having different radii. The probability distribution of patterns is defined characterizing the probabilistic weight of different filaments. The most probable filament radius and filament number are found, being in good agreement with experiment.

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There exists a well established analogy between the evolution equations describing optical phenomena and hydrodynamic equations for compressible viscous liquid [1–3]. The Fresnel number for optical systems plays the same role as the Reynolds number for fluids. In optical systems, with increasing Fresnel number, there appears chaotic behaviour similar to fluid turbulence arising with increasing Reynolds number. Because of this analogy, one terms as the *optical turbulence* [1–3] the chaotic phenomena occurring in high-Fresnel-number optical systems.

For an optical cavity of the standard cylindrical shape of radius R and length L, the Fresnel number is $F \equiv \pi R^2 / \lambda L$, with λ being the optical wavelength. Physical processes accompanying the route from the regular behaviour to the turbulent regime, arising with increasing Fresnel numbers, are practically the same in different types of active optical media, such as lasers and photorefractive crystals.

When Fresnel numbers are very small, $F \ll 1$, there exists the sole transverse mode almost uniformly filling the optical cavity. At Fresnel numbers around $F \sim 1$, several transverse modes emerge seen as bright spots in the transverse cross-section. These spatial structures are regular both in space and in time, since in space they form ordered geometric arrays, as polygons, and because in time they are either stationary or oscillating periodically. These transverse structures are imposed by the cavity geometry and correspond to the empty-cavity Gauss-Laguerre modes. The appearance of such regular geometric structures is well understood theoretically, with the theory being in good agreement with experiments both for lasers [4–9] as well as for photorefractive crystals [10–12]. For Fresnel numbers up to $F \approx 5$, the number of regular modes is proportional to F^2.

The transition from the regular behaviour to a turbulent state occurs around the Fresnel number $F \approx 10$. For Fresnel numbers $F > 15$, the arising spatial structures are principally different from those defined by the empty-cavity Gauss-Laguerre modes. The related modal description becomes no longer relevant and the boundary conditions do not play the role any more. In the medium, there appears a large number of filaments stretched along the cavity axis. In the transverse cross-section, they are distributed chaotically. The correlation length is much shorter than the mean distance between filaments, so that they radiate independently of each other, aperiodically flashing in time. The number of filaments, contrary to the regular regime, is proportional to F. This chaotic filamentation has been observed in both photorefractive crystals [10–12], e.g. in $Bi_{12}SiO_{20}$, as well as in many lasers, such as Ne, Tl, Pb, N_2, and N_2^+ vapor lasers [13–17], CO_2 and dye lasers [18–26]. The specific random properties of the observed filamentation, such as chaotic distribution of filaments in space, the absence of correlations between them, and their aperiodic flashing, are analogous to the properties of turbulent fluids [1–3]. Because of this analogy of the chaotic photon filamentation in high-Fresnel-number optical media with turbulent phenomena in high-Reynolds-number viscous fluids, one commonly calls [18–26] the former the optical turbulence or, because of the formation of bright filaments with a high density of photons, it can be called the *turbulent photon filamentation*.

Contrary to the low-Fresnel number case, with regular optical structures, for which a good theoretical understanding has been achieved [4–12], for the high-Fresnel-number regime of the turbulent photon filamentation, there has been no persuasive theory suggested. This problem was addressed in Refs. [27–30]. However, because of the complexity of the phenomenon, only some oversimplified stationary models were considered. In the present paper, the description of the phenomenon of turbulent photon filamentation is suggested, based on realistic evolution equations. It has become possible to solve now this problem owing to the recently developed method for treating nonlinear differential equations, called the *scale separation approach* [31–33].

For the system of N resonant atoms interacting with electromagnetic field, we use the standard dipole approximation. The evolution equations to be considered are those for the averages of atomic operators,

$$u(\vec{r}, t) \equiv\; < \sigma^-(\vec{r}, t) > \;, \qquad s(\vec{r}, t) \equiv\; < \sigma^z(\vec{r}, t) > \;, \tag{1}$$

where σ^- is a transition operator and σ^z is the population-difference operator. Equations for the quantities (1) can be easily obtained following the usual way, by writing the Heisenberg equations of motion, eliminating the field variables, and employing the standard semiclassical and Born approximations. All this machinery is well known and can be found, for instance, in the book [34]. In order to present resulting equations in a compact form, it is convenient to introduce the notation $f \equiv f_0 + f_{rad}$ for an effective field acting on an atom. This field consists of the term $f_0 \equiv -i\vec{d} \cdot \vec{E}_0$, due to the cavity seed field, and of the term

$$f_{rad}(\vec{r}, t) \equiv\; -\frac{3}{4}\, i\gamma\rho\, \int \left[\varphi(\vec{r} - \vec{r}\,')\, u(\vec{r}\,', t) - \vec{e}_d^{\,2}\, \varphi^*(\vec{r} - \vec{r}\,')\, u^*(\vec{r}\,', t) \right]\, d\vec{r}\,' \tag{2}$$

describing the effective interaction of atoms through the common radiation field. Here ρ is the density of atoms, $\vec{d} \equiv d_0 \vec{e}_d$ is the transition dipole, and $\varphi(\vec{r}) \equiv \exp(ik_0|\vec{r}|)/k_0|\vec{r}|$, with $k_0 \equiv \omega_0/c$ and $\gamma \equiv 4k_0^3 d_0^2/3$. The cavity seed field selecting a longitudinal propagating mode, is taken, as usual, in the form of plane waves, $\vec{E}_0 = \vec{E}_1 e^{i(kz-\omega t)} + \vec{E}_1^* e^{-i(kz-\omega t)}$. The resulting equations are

$$\frac{du}{dt} =\; -(i\omega_0 + \gamma_2)u + sf\;, \qquad \frac{ds}{dt} =\; -2(u^*f + f^*u) - \gamma_1(s - \zeta)\;,$$

$$\frac{d|u|^2}{dt} = -2\gamma_2|u|^2 + s(u^*f + f^*u) , \tag{3}$$

where ω_0 is the transition frequency, γ_1 and γ_2 are the longitudinal and transverse relaxation widths, respectively, and ζ is the pumping parameter.

We consider a cylindrical cavity, typical of lasers, with radius R and length L. For the case of high Fresnel numbers $F \gg 10$, the cavity can house several transverse modes. Therefore, it is reasonable to look for solutions of Eqs. (3) in the form of a bunch of N_f filaments, so that

$$u(\vec{r},t) = \sum_{n=1}^{N_f} u_n(\vec{r},t)\Theta_n(x,y)\,e^{ikz} , \qquad s(\vec{r},t) = \sum_{n=1}^{N_f} s_n(\vec{r},t)\Theta_n(x,y) , \tag{4}$$

where $\Theta_n(x,y) = \Theta(b_n^2 - (x-x_n)^2 + (y-y_n)^2)$ is a unit-step function; the pair x_n and y_n defines the axis of a filament; and b_n is the radial distance from the filament axis, at which the solutions decrease by an order of magnitude. If the filament profile is approximately of normal law $\exp(-r^2/2r_n^2)$, with r_n being the filament radius, then $b_n = \sqrt{2\ln 10}\,r_n$. The filaments do not interact with each other, because of which they cannot form a regular space structure, and the locations of their axes in the transverse cross-section are random. The absence of interactions between filaments is due to the fact that the kernel $\varphi(\vec{r})$, playing the role of a Green function in the effective interaction field (2), fastly oscillates and diminishes with increasing r. Let us stress that the presentation of solutions in the form of a bunch of filaments (4) does not exclude the possible case of a sole filament almost uniformly filling the whole sample. This is because the number of filaments and their radii will be defined in a self-consistent way. Substituting the form (4) into Eqs. (3), we obtain a system of equations for u_n, s_n, and $|u_n|^2$. To simplify these equations, we employ the mean-field approximation for the averages

$$u(t) \equiv \frac{1}{V_n}\int_{V_n} u_n(\vec{r},t)\,d\vec{r} , \qquad s(t) \equiv \frac{1}{V_n}\int_{V_n} s_n(\vec{r},t)\,d\vec{r} , \tag{5}$$

and for the similarly defined $|u(t)|^2$, where the integration is over the volume $V_n \equiv \pi b_n^2 L$ of a cylinder enveloping a filament. For the simplicity of notation, the index n labelling filaments is dropped from the left-hand sides of Eqs. (5). In this way, Eqs. (3) are transformed to the system of equations

$$\frac{du}{dt} = -(i\Omega + \Gamma)u - is\vec{d}\cdot\vec{E}_1\,e^{-i\omega t} ,$$

$$\frac{ds}{dt} = -4g\gamma_2|u|^2 - \gamma_1(s-\zeta) - 4\,\mathrm{Im}\left(u^*\vec{d}\cdot\vec{E}_1\,e^{-i\omega t}\right) , \tag{6}$$

$$\frac{d|u|^2}{dt} = -2\Gamma|u|^2 + 2s\,\mathrm{Im}\left(u^*\vec{d}\cdot\vec{E}_1\,e^{-i\omega t}\right)$$

for the functions (5). Here the notations for the collective frequency $\Omega \equiv \omega_0 + g'\gamma_2 s$ and for the collective width $\Gamma \equiv \gamma_2(1-gs)$ are used, with the effective coupling parameter

$$g \equiv \frac{3\gamma\rho}{4\gamma_2 V_n}\int_{V_n}\frac{\sin[k_0|\vec{r}-\vec{r}\,'| - k(z-z')]}{k_0|\vec{r}-\vec{r}\,'|}\,d\vec{r}\,d\vec{r}\,' , \tag{7}$$

and with g' defined as is Eq. (7) except for sine being replaced by cosine.

Equations (6) describe the evolution of a filament. These evolution equations can be analysed by using the scale separation approach [31–33]. For applying the latter, we, first, take into account the existence of usual small parameters $\gamma_1/\omega_0 \ll 1$ and $\gamma_2/\omega_0 \ll 1$, and also we consider the quasiresonance case, when $|\Delta|/\omega_0 \ll 1$, with the detuning $\Delta \equiv \omega - \omega_0$. Then the functional variable u in Eqs. (6) is to be classified as fast, as compared to the slow variables s and $|u|^2$. The latter are quasi-invariants for the fast variable. Following the scale separation approach [31-33], we solve the equation for the fast variable u, which is straightforward when s is a quasi-invariant. The found solution for u has to be substituted into the right-hand sides of the equations for the slow variables s and $|u|^2$, with averaging these right-hand sides over explicitly entering time of fast oscillations. This procedure, complimented by the transformation

$$w \equiv |u|^2 - \alpha s^2 , \qquad \alpha \equiv \frac{|\vec{d}\cdot\vec{E}_1|^2}{(\omega-\Omega)^2+\Gamma^2} , \tag{8}$$

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where α is assumed to be small, $\alpha \ll 1$, leads to the equations

$$\frac{ds}{dt} = -4g\gamma_2 w - \gamma_1 (s - \zeta) , \qquad \frac{dw}{dt} = -2\gamma_2 (1 - gs)w \qquad (9)$$

for the slow variables.

The properties of solutions to Eqs. (9) essentially depend on the value of the effective coupling g defined in Eq. (7). From this definition, it is evident that the value of g depends on the characteristics of the considered filament, in particular, on the filament radius r_n which enters Eq. (7) through the radius b_n of the enveloping cylinder. It is convenient to introduce the dimensionless parameter $\beta \equiv \pi b_n^2 / \lambda L$ and to consider $g = g(\beta)$ as a function of this parameter in the domain $0 \le \beta \le F$. Thus, each filament can be characterized by its radius or, equivalently, by the related parameter β. The bunch of filaments parametrized by β forms the solutions (4). The stability properties of filaments with different values of β are different, which can be described by defining the probability distribution of patterns corresponding to filaments with β varying in the interval $[0, F]$. The latter can be done as follows. Any system of evolution equations, like Eqs. (9), can always be presented in the normal form $dy/dt = v$, in which $y = \{y_i\}$, and $v = \{v_i\}$ is a velocity field. In the case of Eqs. (9), we have $y_1 \equiv s$, $y_2 \equiv w$.

As is known from statistical mechanics, a probability p is connected with entropy S by the relation $p \sim e^{-S}$. In the nonequilibrium case, it is more appropriate to count the entropy $S(t)$, which is a function of time, from its initial value $S(0)$, thus, considering the entropy variation $\Delta S(t) \equiv S(t) - S(0)$. The entropy is defined as the logarithm of an elementary phase volume, $S(t) = \ln |\Delta\Phi(t)|$, where, in the nonequilibrium case, $\Delta\Phi(t) \equiv \prod_j \delta y_j(t)$. The elementary phase volume can be expressed as $\Delta\Phi(t) = \prod_i \sum_j M_{ij}(t)\delta y_j(0)$ through the elements $M_{ij}(t) \equiv \delta y_i(t)/\delta y_j(0)$ of the multiplier matrix $\hat{M}(t) = [M_{ij}(t)]$. Hence, the entropy variation is $\Delta S(t) = \mathrm{Tr}\hat{L}$, with the matrix $\hat{L} = [L_{ij}]$ being composed of the elements $L_{ij} \equiv \ln M_{ij}$. By variating the evolution equations, one gets the equation $d\hat{M}/dt = \hat{J}\hat{M}$ for the multiplier matrix $\hat{M}(t)$, where $\hat{J} = [J_{ij}]$ is the Jacobian matrix with the elements $J_{ij} \equiv \delta v_i/\delta y_j$. In this way, the probability $p \sim e^{\Delta S}$ acquires the form $p \sim \exp\{-\mathrm{Tr}\hat{L}\}$. From the equation for the multiplier matrix, one has $d\mathrm{Tr}\hat{L}/dt = \mathrm{Tr}\hat{J}$, so that the entropy variation is $\Delta S(t) = \int_0^t \Lambda(t')dt'$, where $\Lambda \equiv \mathrm{Tr}\hat{J}$ is called the contraction rate [35]. Therefore, for the probability distribution of patterns, labelled by a parameter β, one has

$$p(\beta, t) = \frac{1}{Z(t)} \exp\left\{ - \int_0^t \Lambda(\beta, t') \, dt' \right\} , \qquad (10)$$

where Z is the normalization factor. As is clear, that pattern is preferred over the other which possesses a higher probability weight. This implies, because of the form (10), that the most preferable pattern is that for which the contraction is minimal. At the initial stage of pattern formation, one may write $p \sim \exp\{-\Lambda(\beta, 0)t\}$. Hence, the most probable pattern to be formed is that corresponding to the minimal contraction rate $\Lambda(\beta, 0)$.

Considering the evolution equations (9), we keep in mind the standard way of exciting laser media by means of a nonresonant pumping described by the pumping parameter $\zeta > 0$. No resonant fields are involved, so that $s(0) < 0$. It is easy to calculate the contraction rate $\Lambda(\beta, 0)$ corresponding to Eqs. (9), which is $\Lambda(\beta, 0) = -\gamma_1 - 2\gamma_2(1 - gs_0)$, where $s_0 \equiv s(0)$. The minimum of this contraction rate is provided by the maximum of $g = g(\beta)$. The maximum of $g(\beta)$, with $g(\beta)$ given by Eq. (7), occurs at $\beta = 0.96$. From here we find the most probable filament radius

$$r_f = 0.26\sqrt{\lambda L} .$$

The most probable number of filaments can be evaluated from the normalization integral $\frac{1}{V} \int s(\vec{r}, t)d\vec{r} = \zeta$, where the integration runs over the whole volume of the sample, $V = \pi R^2 L$. Considering this normalization integral for the stage when the filaments have already been formed and the population difference inside a filament of radius r_f has reached a value close to ζ, we obtain

$$N_f = \left(\frac{R}{r_f}\right)^2 = 4.71F .$$

The number of filaments is proportional to the Fresnel number F, which is common for the turbulent photon filamentation.

Calculating the filament radius r_f for the variety of experiments with different lasers, we find that the theoretical values of r_f are in perfect agreement with all experimental data available. Thus, for Ne, Tl, Pb, N_2, and N_2^+ laser [13–17], we find $r_f \approx 0.01$ cm and for CO_2 and dye lasers [18–24], we have $r_f \approx 0.08$ cm and $r_f \approx 0.01$ cm, respectively.

In conclusion, the solution for the problem of turbulent photon filamentation occurring in resonant media at high Fresnel numbers is suggested. The consideration is based on realistic evolution equations for resonant atoms under

conditions typical of lasers. The solutions to these equations have the form corresponding to a bunch of filaments with different radii. The probability distribution of filament radii is found self-consistently from the evolution equations. The results are in good agreement with all experiments on the turbulent photon filamentation in laser media [13–24].

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